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right ; and again *MF*, *MH* are equal. Therefore (Eu. I. 26) so are the angles *FML*, *HMK* equal. Wherefore the angle *HMK* makes two right angles with the angle *HML*, since with this the angle *FML* (Eu. I. 13) makes two right angles. Therefore (Eu. I. 14) *KML* will be one continuous straight line, consequently the common perpendicular of the aforesaid straights *AX*, *BX*. Quod erat etc.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from February Number.]

THE CONSTRUCTION OF THE PRIMITIVE GROUPS.

We have shown that all the intransitive and the non-primitive groups of a given degree, may be made to depend upon groups of a lower degree. We shall soon prove a similar property of the primitive groups.

It must however not be inferred that this will solve, in a satisfactory manner, the problem of constructing all the groups of a given degree. The elementary methods to which we have confined ourselves require a large number of trials if the degree is large. Some briefer methods will be given later but even these will only tend to make the construction of all the groups of a given degree practical for somewhat larger degrees.

It is not difficult to give general theorems which include all the groups of a given type, as, for instance, the theorem at the end of our discussion of the construction of the non-primitive groups ; but new types arise continually and no non-tentative method by means of which all the groups of any given degree may be found has yet been published.

We proceed to prove some theorems which apply to all transitive groups but are especially useful in the construction of primitive groups. Unless the contrary is stated the symbols G, g, and n will respresent respectively the group under consideration, its order and its degree.

Let us consider the transitive group G which contains the letters $a_1, a_2, \ldots\ldots, a_n$. The substitutions of G which do not contain a_1 (*i. e.*, those which replace a_1 by itself) may be represented by

$$s_1,\ s_2,\ \ldots\ldots,\ s_r \equiv G_1.$$

As every group must contain the identical substitution if the number of its letters is finite and this is the only kind we are considering now, the minimum value of r is unity.

Since G is transitive it must contain some substitution s_{r+1} which replaces a_1 by a_2. We desire to find all the substitutions of G which have this property. If s_k is such a substitution then will

$$s_k s_{r+1}^{-1}$$

belong to the first line, since s_k replaces a_1 by a_2 and s_{r+1}^{-1} replaces a_2 by a_1 $s_k s_{r+1}^{-1}$ must leave a_1 unchanged. Hence we have the equations

$$s_k s_{r+1}^{-1} = s_\alpha \qquad (\alpha = 1, 2, \ldots\ldots r)$$

$$s_k = s_\alpha \, s_{r+1}.$$

Since the condition expressed by the last equation is sufficient as well as necessary it follows that there are just r different substitutions in G, which transform a_1 into a_2. Similarly there are exactly r substitutions in G which replace a_1 by a_3, etc. From this we see that the number of substitutions which replace a_1 by itself is equal to the number of those which replace a_1 by any other letter of G. We have imposed no condition upon a_1 which is not satisfied by each of the other letters so that the property which we have proved in regard to a_1 belongs to all the letters. That r has the same value for each of the letters follows from the following considerations :

If the substitutions of G which do not contain a_α are

$$s_1^1, . s_2^1 \ldots\ldots s_{r'}^1 \equiv G_2,$$

then will

$$s_{r+1} G_2 s_{+1}^{-1} = r^1 \text{ substitutions of } G \text{ which do not contain } a_1 \text{ and}$$

$$s_{r+1}^{-1} G_1 s_{r+1} = r \text{ substitutions of } G \text{ which do not contain } a_2.$$

From the first of these two equations we have $r' \leqq r$ and from the second $r \leqq r'$, hence $r' = r$. Similar remarks clearly apply to all the letters of G. We may embody the results at which we have arrived in the following

THEOREM: *The number of substitutions* (r) *of any transitive group* (G), *which do not contain any given letter, is equal to the number of substitutions which replace a letter by any required other letter of the group.*

Corollary I. $g = nr$, *i. e. the order of any transitive group is a multiple of its degree.*

Corollary II. The average number of letters in all the substitutions of a transitive group of degree n is $n-1$.*

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*Since every intransitive group may be resolved into transitive constituent groups whose separate elements enter an equal number of the substitutions of the intransitive group, the general statement of this corollary is as follows: *The average number of letters in all the substitutions of any group is* $n-\alpha$, *n being the degree of the group and* α *the number of its transitive constituents.*

The last corollary may be proved as follows: Since G contains only $g \div n$ substitutions that do not involve a_α it must contain $g - g/n = \frac{n-1}{n}g$ that involve a_α. Hence all the g substitutions of G contain $n \times \frac{n-1}{n}g = (n-1)g$ letters.

From this corollary we may directly derive the following:

Corollary III. Every transitive group contains at least $n-1$ substitutions of the n^{th} degree.

Corollary IV. If the order of a transitive group exceeds its degree it must contain substitutions of a lower than the n^{th} degree and hence n conjugate subgroups $G_1, G_2, \ldots\ldots, G_n$ whose degree is at most $n-1$. These n subgroups need not all be distinct.

We may divide the primitive groups into two classes. (1) Those whose order is equal to their degree—the *regular* primitive groups—and (2) those whose order is b times their degree, where b is a positive integer larger than 1.

We proceed to consider the first one of these classes. Since the average number of letters in its substitutions is $n-1$ it must contain $n-1$ substitutions of the n^{th} degree, *i. e.* all its substitutions except unity are of the n^{th} degree.

If any one of these $n-1$ substitutions consists of two or more cycles all of these cycles will be of the same order, *i. e.* they will all contain the same number of letters, otherwise some power of this substitution would at the same time differ from identity and not contain all the letters of the group.

We proceed to prove the following

THEOREM: *Whenever a regular group contains a substitution (s) which contains more than one cycle it is non-primitive.*

Let $s = a_1 a_2 \ldots\ldots a_r, b_1 \ldots\ldots\ldots\ldots\ldots$ Some substitution of G (s_1) replaces a_1 by b_1. If we transform s with respect to s_1 we have

$$s_1^{-1} s\, s_1 = b_1 b_2 \ldots\ldots b_r \ldots\ldots$$

If we assume that

$$b_\alpha = a_\beta \qquad (\alpha,\ \beta \overline{\gtrless} r)$$

we have as a consequence that s_1 replaces a_α by a_β. This is also done by $s^{\beta-\alpha}$. Since only one substitution of G can perform this operation we have as a second consequence of the given assumption

$$s_1 = s^{\beta-\alpha}.$$

The latter of these transforms the cycle $a_1 a_2 \ldots\ldots a_r$ into itself and the former does not, the given assumption is therefore untenable and the cycle of b's must be distinct from the cycle of a's.

If these a's and b's do not include all the letters of G there must be some substitution of G (s_2) which replaces a_1 by some new letter c_1. We now derive the substitution

$$s_2^{-1} s \; s_2 = c_1 c_2 \ldots\ldots c_2 \ldots\ldots$$

We have already proved that these c's are all different from the a's. It remains to show that they do not include any b.

From $$c_\alpha = b_\beta$$

it would follow that s_2 replaced a_α by b_β and therefore that

$$s_2 = s^{\beta-\alpha} s_1.$$

This is impossible since the second member replaces the a's by the b's and the first replaces a_1 by c_1.

Continuing in this manner we must finally exhaust the letters of G and obtain the l distinct cycles

$$a_1 a_2 \ldots\ldots a_r, \; b_1 b_2 \ldots\ldots b_r, \; \ldots\ldots, \; l_1 l_2 \ldots\ldots l_r$$

where $lr = n$, the degree of G.

We proceed to prove that these cycles may be used as systems of non-primitivity. This is, of course, included in the proof that the substitutions composed of these cycles

$$a_1 a_2 \ldots\ldots a_r. \; b_1 b_2 \ldots\ldots b_r \; \ldots\ldots \; l_1 l_2 \ldots\ldots l_r \equiv t$$

is transformed into itself by all the substitutions of G.

Let s_α represent any substitution of G; we desire to prove that

$$s_\alpha^{-1} t s_\alpha = t.$$

If s_α replaces c_γ by b_β we have

$$s_\alpha = s_2^{-1} s^{\beta-\gamma} s_1.$$

The second member replaces $c_{\gamma+\rho}$ by $b_{\beta+\rho}$ where ρ satisfies the congruence

$$\gamma+\rho, \; \beta+\rho \equiv \delta \pmod{r}, \; (\delta = 1, 2, \ldots\ldots r).$$

Hence s_α must replace the c's in order by the b's in order. Since similar remarks apply to all the cycles it follows that s_α which is any substitution of G transforms t into itself and our theorem is proved.

By starting with the different cycles of G which contain the same letter we obtain different systems of non-primitivity for the same group.*

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*Cf. Jordan, Traite des Substitutions, §75; and Netto, Theory of Substitutions (American Edition), §98.

From the last theorem we see that a regular group cannot be primitive unless it is generated by a single cycle involving a prime number of letters. Since such a group must be primitive we have the following

THEOREM : *The regular primitive groups and the prime numbers have a* 1,1 *correspondence ; i. e. for each prime number there is one regular primitive group and for each regular primitive group there is one prime number.*

[To be Continued.]

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THE CENTROID OF AREAS AND VOLUMES.

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[Continued from February Number.]

II. VOLUMES. Let the density vary as $x^{h-1}y^{k-1}z^{l-1}$. Then

$$\bar{x}=\frac{\iiint x^h y^{k-1} z^{l-1}dxdydz}{\iiint x^{h-1}y^{k-1}z^{l-1}dxdydz},\quad \bar{y}=\frac{\iiint x^{h-1}y^k z^{l-1}dxdydz}{\iiint x^{h-1}y^{k-1}z^{l-1}dxdydz},$$

$$\bar{z}=\frac{\iiint x^{h-1}y^{k-1}z^l dxdydz}{\iiint x^{h-1}y^{k-1}z^{l-1}dxdydz}.$$

$$\therefore \bar{x}=\dfrac{\dfrac{a^{h+1}b^kc^l}{\frac{8}{(2m+1)(2n+1)(2p+1)}}\;\dfrac{\Gamma\left\{\frac{h+1}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n+1)\right\}\Gamma\left\{\frac{l}{2})2p+1)\right\}}{\Gamma\left\{\frac{h+1}{2}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}{\dfrac{a^hb^kc^l}{\frac{8}{(2m+1)(2n+1)(2p+1)}}\;\dfrac{\Gamma\left\{\frac{h}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n=1)\right\}\Gamma\left\{\frac{l}{2}(2p+1)\right\}}{\Gamma\left\{\frac{h}{3}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}$$